The Sum-Eccentricity Energy Of A Graph

Mohammad Issa Sowaity and B. Sharada

Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru - 570 006, INDIA
E-mail address: mohammad_d2007@hotmail.com

Department of Studies in Computer Scince, University of Mysore, Manasagangotri, Mysuru - 570 006, INDIA
E-mail address: sharadab21@gmail.com

Abstract. In this paper, we introduce the concept of the sum-eccentricity matrix $S_e(G)$ of a graph G and obtain some coefficients of the characteristic polynomial $P(G,\lambda)$ of the sum-eccentricity matrix of G. We also introduce the sum-eccentricity energy $ES_e(G)$ of a graph G. Sum-eccentricity energies of some well-known graphs are obtained. Upper and lower bounds for $ES_e(G)$ are estblished. It is shown that if the sum-eccentricity energy of a graph is rational then it must be an even.

Key words and phrases. *Distance in graphs, Sum-eccentricity matrix, Sum-eccentricity eigenvalues, Sum-eccentricity energy of a graph.*

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1. Introduction

In this paper, all graphs are assumed to be finite connected simple graphs. A graph $G=(V,E)$ is a simple graph, that is, having no loops, no multiple and directed edges. As usual, we denote n to be the order and m to be the size of the graph G. For a vertex $v \in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertecies that are adjacent to v and the closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. The degree of a vertex v in G is $d(v)=|N(v)|$. The distance $d(u,v)$ between any two vertices u and v in a graph G is the length of the shortest path connecting them. The eccevtricity of a vertex $v \in G$ is $e(v) = \max\{d(u,v): u \in V(G)\}$. The radius of G is $r(G) = \min\{e(v): v \in V(G)\}$ and the diameter of G is $D(G) = \max\{e(v): v \in V(G)\}$. Hence $r(G) \le e(v)(\le D(G),$ for every $v \in V(G)$. A vertex v in a connected graph G is central if $e(v) = r(G)$, while a vertex v in a connected graph G is peripheral vertex if $e(v) = D(G)$. A graph G is called self centered graph if $e(v) = r(G) = D(G)$. The girth of a graph G is the length of the shortest cycle contained in the graph and denoted by $g(G)$. All the defnitions and terminologies about the graph in this paragraph available in [9].

The concept energy of a graph introduced by I. Gutman [8], in (1978). Let G be a graph with n vertices and m edges and let $A(G) = (a_{ij})$ be the adjacency matrix of G, where

$$a_{ij} = \begin{cases} 1, \text{ if } v_i v_j \in E, \\ 0, \text{ otherwise.} \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix $A(G)$ assumed in a non-increasing order, are the eigenvalues of a graph G[10]. Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_t$, for $t \le n$ be the distinct eigenvalues of G with multiplicities $m_1, m_2, \ldots, m_t$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Sp(G) = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{bmatrix}$$

As A is real symmetric with zero trace, the eigenvalues of G are real with sum equal to zero [3]. The energy $E(G)$ of a graph G is defined to be the sum of the absolute values of the eigenvalues of G[8], i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy we refer to [5, 7, 10] and the references therein.

C. Adiga et. al. [2], have defined the maximum degree energy $E_M(G)$ of a graph G which depends on the maximum degree matrix $M(G)$ of G. Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$. Then the maximum degree matrix $M(G) = (d_{ij})$ of a graph G defined as

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$$d_{ij} = \begin{cases} \max\{d(v_i), d(v_j)\}, \textit{if } v_iv_j \in E, \\ 0, \textit{otherwise}. \end{cases}$$

As $M(G)$ is real symmetric with zero trace, then the eigenvalues of G being real with sm equal to zero.

Ahmed M. Naji et. al. [3], have defined the concept of maximum eccentricity matrix $M_e(G)$ of a connected graph G. They obtained the maximum eccentricity energy $EM_e(G)$ of a graph depends on the maximum eccentricity matrix. Let G be a simple connected graph with n vertices $v_1, v_2, \ldots, v_n$ and let $e(v_i)$ be the eccentricity of a vetex $v_i, i = 1, 2, ..., n$ The maximum eccentricity matrix of G defined as $M_e(G) = (e_{ij})$, where

$$e_{ij} = \begin{cases} \max\{e(v_i), e(v_j)\}, \textit{if } v_iv_j \in E, \\ 0, \textit{otherwise}. \end{cases}$$

Motivated by those papers, we introduce the concept of the sum-eccentricity matrix $S_e(G)$ of a graph G and obtain some coefficients of the characteristic polynomial $P(G, \lambda)$ of the sum-eccentricity matrix of G. We also introduce the sum-eccentricity energy $ES_e(G)$ of a graph G. Sum-eccentricity energies of some well-known graphs are obtained. Upper and lower bounds for $ES_e(G)$ are estblished. It is shown that if the sum-eccentricity energy of a graph is rational then it must be an even.

2. THE SUM-ECCENTRICITY ENERGY OF GRAPHS

Definition 2.1. Let G be a graph with n vertices. Then the sum-eccentricty matrix of a graph G denoted by $S_e(G)$, is defined as $S_e(G) = (s_{ij})$, where

$$s_{ij} = \begin{cases} e(v_i) + e(v_j), \textit{if } v_iv_j \in E, \\ 0, \textit{otherwise}. \end{cases}$$

The characteristic polynomial of the sum-eccentricity matrix $S_e(G)$ is defined by

$$P(G, \lambda) = \det(\lambda I - S_e(G)),$$

Where I is the unt matrix of order n The eigenvalues of the sum-eccentricity matrix $S_e(G)$ are the roots of the charecteristic polynomial of

G.

Since $S_e(G)$ is real symmetric with zero trace, its eigenvalues must be realwith sum equal to zero, i.e., $trace(S_e(G))=0$. We lable the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ in a non-increasing manner $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$. The sum-eccentrcity energy of a graph G is denoted by $ES_e(G)$ and is defined as the summation of the absolute value of the eigenvalues

$$ES_e(G)=\sum_{i=1}^{n}|\lambda_i|.$$

The following examples explain the concept.

Example 2.2. Let G_1 be the graph as in figure 1.

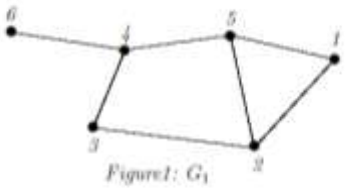

Figure1: G_1

Then the sum-eccentricity matrix of G_1 is

$$S_e(G_1)=\begin{bmatrix}0&6&0&0&5&0\\6&0&5&0&5&0\\0&5&0&4&0&0\\0&0&4&0&4&5\\5&5&0&4&0&0\\0&0&0&5&0&0\end{bmatrix}$$

The charracteristic polynomial of $S_e(G_1)$ is

$$P(G_1,\lambda)=\lambda^6-168\lambda^4-300\lambda^3+4952\lambda^2+7500\lambda-15625.$$

The sum-eccentricity eigenvalues of G_1 are

$$\lambda_1=12.54,\lambda_2=5.4884,\lambda_3=1.2211,\lambda_4=-2.8779,\lambda_5=-6.6336,\lambda_6=-9.7383.$$

The sum-eccentricity energy of G_1 is

$$ES_e(G_1)=38.499.$$

Example 2.3. Let G_2 be the K_5graph.

Then the sum-eccentricity matrix of G_2 is

$$S_e(G_2)=\begin{bmatrix}0&2&2&2&2\\2&0&2&2&2\\2&2&0&2&2\\2&2&2&0&2\\2&2&2&2&0\end{bmatrix}$$

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The charracteristic polynomial of $S_e(G_2)$ is

$$P(G_2,\lambda)=\lambda^5-40\lambda^3-16\lambda^2-240\lambda-128=(\lambda+2)^4(\lambda-8).$$

The sum-eccentricity eigenvalues of G_2 are

$$\lambda_1=8,\lambda_2=-2,\lambda_3=-2,\lambda_4=-2,\lambda_5=-2.$$

The sum-eccentricity energy of G_2 is

$$ES_e(G_2)=16.$$

3. BOUNDS FOR SUM-ECCENTRICITY ENERGY AND SUM-ECCENTRICITY EIGENVALUES

We now give the explicit expression for the coefficient c_i of $\lambda^{n-i}(i=0,1,2,3 and\, n)$ in the characteristic polynomial of the sum-eccentricity matrix $S_e(G)$.

Theorem 3.1. Let G be a graph of order n and let

$$P(G,\lambda)=c_0\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+...+c_n,$$

be the charracteristic polynomial of $S_e(G)$. Then

1. $c_0=1.$
2. $c_1=0.$
3. $c_2=-\sum_{i=1,i<j}^{n}(e(v_i)+e(v_j))^2,\ where\ v_iv_j\in E.$
4. $c_3=-2\sum_{\Delta v_iv_jv_k,1\le i<j<k\le n}^{n}(2e(v_i)e(v_j)e(v_k)+e(v_i)^2e(v_j)+e(v_i)^2e(v_k)+e(v_j)^2e(v_i)+$
   $e(v_j)^2e(v_k)+e(v_k)^2e(v_i)+e(v_k)^2e(v_j)).$
5. For $n>1$we have $c_n=(-1)^n\det(S_e(G)).$

Proof. The proof of parts (1) and (2) are similar to the proof in [2].

3. Since

$$c_2=\sum_{1\le i<j\le n}\begin{vmatrix}0 & s_{ij}\\ s_{ji} & 0\end{vmatrix}=\sum_{1\le i<j\le n}0-(s_{ij}s_{ji})=-\sum_{1\le i<j\le n}s_{ij}^{\ 2}$$

and since

$$s_{ij}=\begin{cases}e(v_i)+e(v_j),\ if\ v_iv_j\in E,\\ 0,\ otherwise.\end{cases}$$

Thus $c_2=-\sum_{i=1,i<j}^{n}(e(v_i)+e(v_j))^2,\ where\ v_iv_j\in E.$

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4. We have

$$c_2 = \sum_{1\le i<j<k\le n}^{n} \begin{vmatrix} s_{ii} & s_{ij} & s_{ik} \\ s_{ji} & s_{jj} & s_{jk} \\ s_{ki} & s_{kj} & s_{kk} \end{vmatrix}$$

$$= -2 \sum_{1\le i<j<k\le n}^{n} (s_{ij} s_{ik} s_{jk})$$

$$= -2 \sum_{\Delta v_i v_j v_k, 1\le i<j<k\le n}^{n} [(e(v_i)+e(v_j))(e(v_i)+e(v_k))(e(v_j)e(v_k))]$$

$$= -2 \sum_{\Delta v_i v_j v_k, 1\le i<j<k\le n}^{n} (2e(v_i)e(v_j)e(v_k) + e(v_i)^2 e(v_j) + e(v_i)^2 e(v_k) + e(v_j)^2 e(v_i) +$$

$$e(v_j)^2 e(v_k) + e(v_k)^2 e(v_i) + e(v_k)^2 e(v_j)).$$

5. We have $c_k = (-1)^k \sum_{k=1}^{n} (all\ k \times k\ principle\ \min ors)$

hence $c_n = (-1)^n \det(S_e(G))$.

Example 3.2. In the graph G_1 in figure 1, the coefficient c_2 of λ^4 in the characteristic polynomialof $S_e(G_1)$ is equal to

$$-\sum_{i=1, i<j}^{n} (e(v_i)+e(v_j))^2, \ where\ v_i v_j \in E$$

$$-[(3+3)^2 + (3+2)^2 + (3+2)^2 + (3+2)^2 + (2+2)^2 + (2+2)^2 + (2+3)^2] = -168$$

Remark 3.3. **a.** The number of terms in c_3 in the above theorem is equal to the number of triangles in the graph.

b. If $g(G) \ne 3$, then $c_3 = 0$.

Theorem 3.4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the sum-eccentricity eigenvalues of a graph G, then

$$\sum_{i=1}^{n} \lambda_i^{\ 2} = -2c_2.$$

Proof. We have

$$\sum_{i=1}^{n} \lambda_i^{\ 2} = trace(S_e^2(G)) = \sum_{i=1}^{n} \sum_{k=1}^{n} s_{ik} s_{ki} = 2\sum_{i=1}^{n} \sum_{i<k}^{n} s_{ik}^2 = 2 \sum_{i=1, i<k}^{n} s_{ik}^2$$

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$$= 2\sum_{i=1, i<k}^{n} (e(v_i) + e(v_k))^2, \ where \ v_i v_k \in E,$$

hence

$$\sum_{i=1}^{n} \lambda_i^{\ 2} = -2c_2.$$

Theorem 3.5. Let $G = K_n$, a complete graph of order $n, n > 1$, then $c_2 = -2n(n-1)$.

Proof. We have $c_2 = -\sum_{i=1, i<j}^{n} (e(v_i) + e(v_j))^2, \ where \ v_i v_j \in E,$

we also have in K_n each $e(v_i) = 1$ so

$$c_2 = -\sum_{i=1}^{n-1} (2+2)^2 i = -4\frac{n(n-1)}{2} = -2n(n-1).$$

Example 3.6. In the graph G_2, the coefficient c_2 of λ^3 in the characteristic polynomialof $S_e(G_2)$ is $-2(5)(4) = -40.$

Corollary 3.7. For the complete graph K_n, we have

$$\sum_{i=1}^{n} \lambda_i^{\ 2} = 4n(n-1).$$

Theorem 3.8. If G is a graph of order n, then for any sum-eccentricity eigenvalue λ_j, we have

$$c_2 \geq \frac{(n-2)\lambda_j^2}{2} - 2n((n-1)^2.$$

Proof. We have

$$trace(S_e^2(K_n)) = 4n(n-1)$$

by Cauchy-Schwartz inequality, we have

$$\sum_{i=1, i\neq j}^{n} \lambda_i^2 \leq (n-1) \sum_{i=1, i\neq j}^{n} \lambda_i^2 = (n-1)(4n(n-1)-\lambda_j^2)$$

so

$$\sum_{i=1, i\neq j}^{n} \lambda_i^2 \leq 4n(n-1)^2 - \lambda_j^2(n-1)$$

$$i.e. \sum_{i=1}^{n} \lambda_i^2 \leq 4n(n-1)^2 - \lambda_j^2(n-1) + \lambda_j^2 = 4n(n-1)^2 - \lambda_j^2(n-2).$$

Using theorem 3.4., we get

$$c_2 \geq \frac{(n-2)\lambda_j^2}{2} - 2n((n-1)^2.$$

Theorem 3.9. We have

$$\sqrt{2\sum_{i=1, i<j}^{n}(e(v_i)+e(v_j))^2 + n(n-1)L^{\frac{2}{n}}} \leq ES_e(G) \leq \sqrt{\frac{2n^2c_2+4n^3(n-1)^2}{n-2}},$$

$where\ v_iv_j \in E,\ L = \prod_{i=1}^{n} \lambda_i\ and\ n > 2$ for the left side of the inequality.

Proof. We have

$$E^2S_e(G) = (\sum_{i=1}^{n} |\lambda_i|)^2$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j|.$$

Using the last inequality in theorem 3.1 and Arithmatic mean, Geometric mean inequality we get

$$E^2S_e(G) = 2\sum_{i=1, i<j}^{n}(e(v_i)+e(v_j))^2 + \sum_{i\neq j} |\lambda_i||\lambda_j|,\ where\, v_iv_j \in E,$$

but

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$$\sum_{i\neq j}|\lambda_i||\lambda_j| = |\lambda_1|(|\lambda_2|+|\lambda_3|+\ldots+|\lambda_n|)$$

$$+|\lambda_2|(|\lambda_1|+|\lambda_3|+\ldots+|\lambda_n|)$$

$$\vdots$$

$$+|\lambda_n|(|\lambda_1|+|\lambda_2|+\ldots+|\lambda_{n-1}|)$$

$$\geq n(n-1)(|\lambda_1||\lambda_2||\lambda_3|\ldots|\lambda_n|)^{\frac{1}{n}}(|\lambda_1|^{n-1}|\lambda_2|^{n-1}|\lambda_3|^{n-1}\ldots|\lambda_n|^{n-1})^{\frac{1}{n(n-1)}}$$

hence

$$\sqrt{2\sum_{i=1,i<j}^{n}(e(v_i)+e(v_j))^2+n(n-1)L^{\frac{2}{n}}} \leq ES_e(G),$$

$where\ v_iv_j \in E\ and\ L=\prod_{i=1}^{n}\lambda_i.$

On the other hand, using the previous theorem we have

$$|\lambda_j| \leq \sqrt{\frac{2c_2+4n(n-1)^2}{n-2}},$$

so

$$\sum_{j=1}^{n}|\lambda_j| \leq \sqrt{\frac{2n^2c_2+4n^3(n-1)^2}{n-2}}, where\ n>2.$$

Theorem 3.10. If the sum-eccentricity energy of a graph G is retional, then it must be an even integer.

Proof. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the sum-eccentricity eigenvalues of a graph G with order n. Then we have $\sum_{i=1}^{n}\lambda_i=0$. Let $\lambda_1,\lambda_2,\ldots,\lambda_r$ be positive, and $\lambda_{r+1},\lambda_{r+2},\ldots,\lambda_n$ arenon-positive. Then,

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$$ES_e(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r).$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are algebraic numbers, so is there sum, and hence must be integer if $ES_e(G)$ is retional. Thus $ES_e(G)$ is an even positive integer if it is rational.

4. THE SUM-ECCENTRICITY ENERGY FOR SOME STANDARD GRAPHS

In this section we investigate the exact values of the sum-eccentricity energy of some well-known graphs.

Theorem 4.1. For the cycle $C_n, n \geq 3,$ is we have

$$c_2 = \begin{cases} -n^3, \; if \; n \, is \, even, \\ -n(n-1)^2, \; if \; n \, is \, odd. \end{cases}$$

Proof. We have $c_2 = -\sum_{i=1, i<j}^{n} (e(v_i) + e(v_j))^2,$

and

$$e(v_i) = \begin{cases} \frac{n}{2}, \; if \; n \, is \, even, \\ \frac{(n-1)}{2}, \; if \; n \, is \, odd, \end{cases}$$

so if n is even $c_2 = -\sum_{i=1, i<j}^{n} (2\frac{n}{2})^2 = -n^3,$

and if n is odd $c_2 = -\sum_{i=1, i<j}^{n} (2\frac{n-1}{2})^2 = -n(n-1)^2,$

thus

$$c_2 = \begin{cases} -n^3, \; if \; n \, is \, even, \\ -n(n-1)^2, \; if \; n \, is \, odd. \end{cases}$$

Theorem 4.1. The sum-eccentricity eigenvalues for the complete graph K_n are -2 and $2(n-1)$ with multiplicities $(n-1)$ and 1 respectively, and the sum-eccentricity energy for K_n is $4(n-1)$.

Proof. We have

$$|\lambda I - S_e(K_n)| = \begin{vmatrix} \lambda & -2 & -2 & \cdots & -2 \\ -2 & \lambda & -2 & \cdots & -2 \\ -2 & -2 & \lambda & \cdots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & -2 & \cdots & \lambda \end{vmatrix}$$

$$= (\lambda+2)^{n-1} \begin{vmatrix} \lambda & -2 & -2 & \cdots & -2 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= (\lambda+2)^{n-1}(\lambda - 2(n-1)).$$

The sum-eccentricity eigenvalues of K_n are $\lambda_1 = 2(n-1), \lambda_2 = -2, \lambda_3 = -2, \ldots, \lambda_n = -2$, i.e., -2 with multiplicity $n-1$ and $2(n-1)$ with multiplicity 1.

Hence $ES_e(K_n) = 4(n-1)$.

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